Homework 8: Solutions to exercises not appearing in Pressley

Math 120A

- (6.2.6) We have a surface patch $\sigma : U \to S$ with first fundamental form $du^2 + f(u)^2 dv^2$. By possibly shrinking U, we can assume that $U = (a,b) \times (c,d)$ where $d - c < 2\pi$. Note that our assertion that $du^2 + f(u)^2 dv^2$ is the first fundamental form implies that $|f(u)| > 0$ everywhere on the surface patch. Ergo without loss of generality $f(u) > 0$ on (a,b). Now, let $g(u) = \int_{u_0}^{u} \sqrt{1 - \dot{f}^2} dt$, where $u_0 \in (a,b)$ is some constant. By assumption $|\dot{f}| < 1$, so the integrand is strictly positive and $g(u)$ is therefore smooth. Furthermore, $\dot{g}^2 + \dot{f}^2 = (1 - \dot{f})^2 + \dot{f}^2 = 1$, so the curve $\gamma(u) = (f(u), 0, g(u))$ is unit speed. We let $\widetilde{\sigma} : U \to \mathbb{R}^2$ be a surface patch with $\widetilde{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(v))$. This is regular surface patch by standard computations for a surface of revolution. Then we consider the map $f(\sigma(U)) \to \widetilde{\sigma(U)}$ which takes $\sigma(u,v) \to \widetilde{\sigma(u,v)}$. Since this map is $\widetilde{\sigma} \circ \sigma^{-1}$, and both surface patches are invertible, it is a local diffeomorphism. Moreover, $\widetilde{\sigma} = f \circ \sigma$, and the first fundamental forms of σ and $\widetilde{\sigma}$ are the same, since the first fundamental form for a surface patch of a surface of revolution obtained from rotating a unit speed curve $\gamma(u) = (f(u), 0, \widetilde{g(u))}$ about the z axis is in general $du^2 + f(u)^2 dv^2$. Ergo the two surfaces $\sigma(u,v)$ and $\sigma(u,v)$ are locally isometric.

- (6.3.9) Let $\sigma(u,v) = (u\cos v, u\sin v, f(u))$. We are interested in finding functions f for which this surface patch is conformal, that is, for which the first fundamental form of the surface patch can be expressed in coordinates as $\lambda(u,v)(du^2 + dv^2)$, where $\lambda(u,v) > 0$ is a smooth function of the domain U of the surface patch. This means we want $\sigma_u \cdot \sigma_u = \lambda = \sigma_v \cdot \sigma_v$ everywhere, and $\sigma_u \cdot \sigma_v = 0$. (In more abstract language, the two partial derivative vectors of σ are orthogonal and have the same length.) We have the following:

$$\begin{aligned} \sigma_u &= (\cos v, \sin v, \dot{f}) \\ \sigma_v &= (-u\sin v, u\cos v, 0) \\ \sigma_u \cdot \sigma_u &= 1 + \dot{f}^2 \\ \sigma_v \cdot \sigma_v &= u^2 \\ \sigma_u \cdot \sigma_v &= 0 \end{aligned}$$

  Ergo we're interested in functions $f(u)$ for which $1 + \dot{f}^2 = u^2$. Note that we must have $u \geq 1$ for this to be true; since our surface patch is presumably defined on an open set we in fact have $u > 1$. Ergo $\dot{f} = \pm\sqrt{u^2 - 1}$, which is smooth on $u > 1$. The integral $\int \sqrt{u^2 - 1} du$ can be solved by hyperbolic or ordinary trig substitution, and works out to

$$f(u) = \pm \frac{1}{2}[u\sqrt{u^2 - 1} - \cosh^{-1}(u)] + c.$$